Guía Semana 10: Sistema masa-resorte

- Regla de asignación de la función f:

$y = A\,\text{sen}(\underbrace{\omega t + \phi}) \quad$ ó $\quad y = A\cos(\omega t + \phi)$

toma valores de 0 hasta 2π para graficar.

$|A| \rightarrow$ Amplitud $\quad \omega =$ rapidez angular $\quad \phi =$ fase inicial

$\omega t_i + \phi = 0 \rightarrow t_i = -\frac{\phi}{\omega}$; $\omega t_f + \phi = 2\pi \rightarrow \frac{2\pi - \phi}{\omega} = t_f$

$T = t_f - t_i = \frac{2\pi - \phi}{\omega} - \left(-\frac{\phi}{\omega}\right) \rightarrow T = \frac{2\pi}{\omega} \rightarrow$ Período

$\rightarrow$ frecuencia $\leftarrow$

$T = \frac{1}{f}$ ó $f = \frac{1}{T}$, $f \rightarrow$ se mide en Hertz. $(\frac{1}{s})$; $f = \frac{\omega}{2\pi}$

↳ nº de ciclos dados en 1 segundo.

$T = \frac{2\pi}{\omega}$, se tiene que $\frac{1}{f} = \frac{2\pi}{\omega} \rightarrow \omega = 2\pi f \rightarrow$ Rapidez angular.

$\omega = \sqrt{\frac{k}{m}}$, $k \rightarrow$ constante del resorte

$m \rightarrow$ masa suspendida en el resorte.

$S \rightarrow$ Distancia desplazada del resorte.

$B \rightarrow$ constante de amortiguamiento. $\rightarrow B\frac{dy}{dt}$

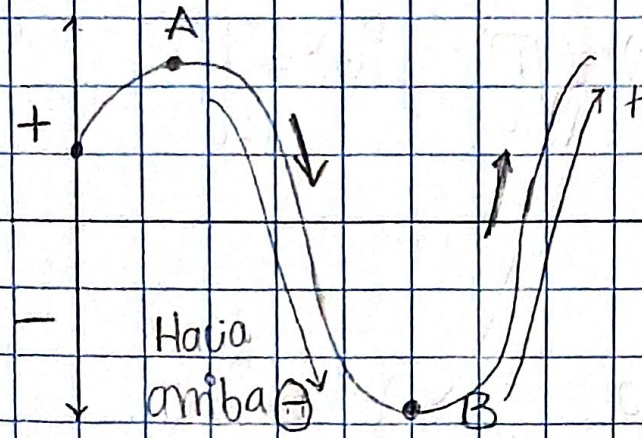
$y_0 \rightarrow$ Posición de equilibrio.

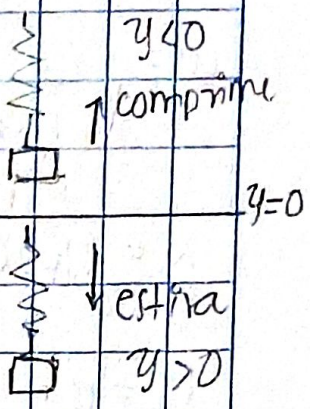
$m\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = f(t) \rightarrow$ con amortiguamiento

$m\frac{d^2y}{dt} + ky = 0$ → sin amortiguamiento. ó movimiento libre no amortiguado.

$y = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \operatorname{sen}\left(\sqrt{\frac{k}{m}}\, t\right)$ ó $y = C_1\cos(\omega t) + C_2 \operatorname{sen}(\omega t)$

$y > 0$ cuando se estira. (Hacia abajo positivo + ↓)
$y < 0$ cuando se comprime. (Hacia arriba negativo − ↑)





$y(t) = A \operatorname{sen}\left(\sqrt{\frac{k}{m}}\, t + \phi\right)$ → donde $A = \sqrt{C_1^2 + C_2^2}$
↳ Amplitud

ϕ → Ángulo de fase → $\tan\phi = \frac{C_1}{C_2}$

1) Se fija una masa de 20 kg a un resorte. Si la frecuencia del movimiento armónico simple es $\frac{2}{\pi}$ oscilaciones por segundo. ¿Cuál es la constante del resorte?, ¿cuál es la frecuencia del movimiento armónico simple si la masa original se reemplaza por una de 80 kg?.

$F = kS \quad k = ?$

$f_1 = \frac{w}{2\pi} \rightarrow w = 2\pi f_1 \rightarrow w = 2\pi\left(\frac{2}{\pi}\right) = 4$

$w = \sqrt{\frac{k}{m_1}} \rightarrow w = \sqrt{\frac{k}{20}} \rightarrow (w)^2 = \left(\sqrt{\frac{k}{20}}\right)^2 \rightarrow (4)^2 = \frac{k}{20}$

$k = (16)(20) = 320 \frac{N}{m}$

$f_2 = ?, \quad m_2 = 80\,kg, \quad k = 320,$

$m\frac{d^2y}{dt^2} + ky = 0 \rightarrow 80y'' + 320y = 0 \rightarrow \frac{80y''}{80} + \frac{320y}{80} = \frac{0}{80}$

$y'' + 4y = 0 \rightarrow$ Ec. auxiliar: $M^2 + 4 = 0$

$M^2 = -4 \rightarrow \sqrt{M^2} = \sqrt{-4} \rightarrow M = \sqrt{+4}\sqrt{-1}$

$M = \pm 2i, \quad \alpha = 0, \quad \beta = 2, \quad M_1 = \alpha + \beta i = +2i, \quad M_2 = \alpha - \beta i = -2i$

Si $y = C_1 e^{\alpha t}\cos(wt) + C_2 e^{\alpha t}\operatorname{sen}(wt)$

$y = C_1 e^{0t}\cos(2t) + C_2 e^{0t}\operatorname{sen}(2t)$

$y = C_1\cos(2t) + C_2\operatorname{sen}(2t)$

Si $\beta = w$ ∴ $w = 2\pi f \rightarrow 2 = 2\pi f \rightarrow f = \frac{2}{2\pi}$

∴ la frecuencia es $\rightarrow f = \frac{1}{\pi}$ Hertz

2) Al fijar un contrapeso de 24 lb al extremo de un resorte, lo estira 4 pulgadas. Deducir la ecuación del movimiento cuando el contrapeso se suelta y parte del reposo desde un punto que está 3 pulgadas arriba de la posición de equilibrio. $F = KS$, $K = ?$

$S = 4$ pulgadas. $W = 24$ lb. $y(0) = -3$ pulgadas ; $y'(0) = 0$

$S = 4 \text{ pulgadas} \left| \frac{1 \text{ pie}}{12 \text{ pulgadas}} \right| = \frac{1}{3}$ pie; $y(0) = -3 \text{ pulgadas} \left| \frac{1 \text{ pie}}{12 \text{ pg.}} \right| = -\frac{1}{4}$ pie

$W = 24$ lb; $m = \frac{W}{g} = \frac{24 \text{ lb}}{32 \text{ pie/s}^2} = \frac{3}{4}$ slug $F = W$

$K = \frac{F}{S} = \frac{24 \text{ lb}}{\frac{1}{3} \text{ pie}} = 72 \frac{\text{lb}}{\text{pie}}$

$m\frac{d^2y}{dt^2} + Ky = 0 \rightarrow \frac{3}{4}y'' + 72y = 0$

Ec. auxiliar $\rightarrow \left[\frac{3}{4}M^2 + 72m\right]\frac{4}{3} = 0 \left[\frac{4}{3}\right]$

$M^2 + 96 = 0 \rightarrow \sqrt{M^2} = \sqrt{-96} \rightarrow M = \pm\sqrt{(4)^2 \cdot 6}$

$M = \pm 4\sqrt{6}$, $\alpha = 0$. $\beta = 4\sqrt{6}$

$y = C_1 e^{0t}\cos(4\sqrt{6}t) + C_2 e^{0t}\operatorname{sen}(4\sqrt{6}t) \rightarrow y = C_1\cos(4\sqrt{6}t) + C_2\operatorname{sen}(4\sqrt{6}t)$

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$\frac{1}{4} = C_1 \cos(4\sqrt{6}(0)) + C_2 \operatorname{sen}(4\sqrt{6}(0)) \rightarrow C_1 = -\frac{1}{4}$

$y' = -4\sqrt{6} C_1 \operatorname{sen}(4\sqrt{6}t) + 4\sqrt{6} C_2 \cos(4\sqrt{6}t)$, si $y'(0) = 0$

$0 = -4\sqrt{6}\left(-\frac{1}{4}\right)\operatorname{sen}(4\sqrt{6}(0)) + 4\sqrt{6} C_2 \cos(4\sqrt{6}(0))$

$0 = 4\sqrt{6} C_2 \rightarrow C_2 = 0$

$\therefore y(t) = -\frac{1}{4}\cos(4\sqrt{6}t)$ → Ec. del movimiento.

3) Formule la Ec. del movimiento si el contrapeso del problema anterior parte de la posición de equilibrio con una velocidad inicial de 2 pies/s hacia abajo.

$k = 72 \frac{lb}{pie}$, $W = 24\ lb$, $m = \frac{3}{4}\ slug$. $y(0) = 0$, $y'(0) = 2 \frac{pie}{s}$

$y(t) = C_1 \cos(4\sqrt{6}t) + C_2 \operatorname{sen}(4\sqrt{6}t)$

$y'(t) = -4\sqrt{6} C_1 \operatorname{sen}(4\sqrt{6}t) + 4\sqrt{6} C_2 \cos(4\sqrt{6}t)$

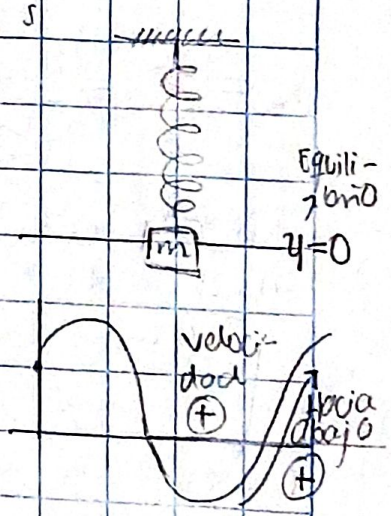
$0 = C_1 \cos(4\sqrt{6}(0)) + C_2 \operatorname{sen}(4\sqrt{6}(0))$

$C_1 = 0$

$2 = -4\sqrt{6}(0)\operatorname{sen}(4\sqrt{6}(0)) + 4\sqrt{6} C_2 \cos(4\sqrt{6}(0))$

$\frac{2}{4\sqrt{6}} = C_2 \rightarrow C_2 = \left(\frac{1}{2\sqrt{6}}\right)\left(\frac{\sqrt{6}}{\sqrt{6}}\right) = \frac{\sqrt{6}}{12}$

$\therefore y(t) = \frac{\sqrt{6}}{12}\operatorname{sen}(4\sqrt{6}t)$ → Ec. del movimiento



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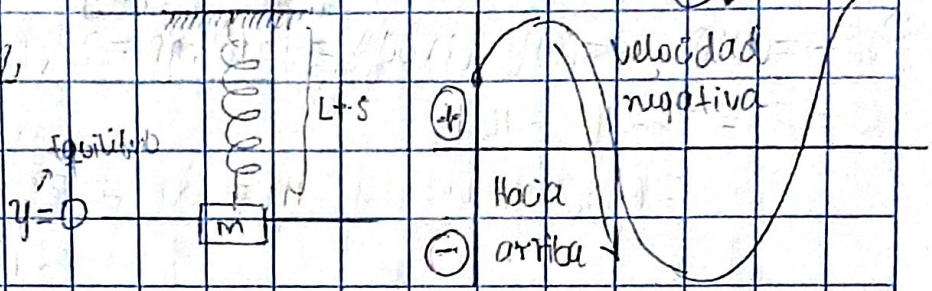
4) Una fuerza de 400 N estira 2m un resorte. Después, al extremo de ese resorte se fija una masa de 50kg y parte de la posición de equilibrio a una velocidad de 10 m/s hacia arriba. Deduzca la Ec. del movimiento.

$F = kS$; $K = ?$, $m = 50kg$,

$S = 2m$, $F = 400N$, $y(0) = 0$

$y'(0) = -10\ m/s$



$$K = \frac{F}{S} = \frac{400N}{2m} = 200\ \frac{N}{m}$$

$$m\frac{d^2y}{dt^2} + ky = 0 \rightarrow 50y'' + 200y = 0 \rightarrow \frac{1}{50}[50y'' + 200y] = \frac{1}{50}[0]$$

Ec. auxiliar $\rightarrow M^2 + 4 = 0 \rightarrow \sqrt{M^2} = \sqrt{-4} \rightarrow M = \sqrt{4}\sqrt{-1}$

$M = \sqrt{4}\sqrt{-1} \rightarrow M = \pm 2i$, $\alpha = 0$, $\beta = 2$

$y(t) = C_1 e^{\alpha t}\cos(2t) + C_2 e^{\alpha t}\operatorname{sen}(2t) = C_1 e^{0t}\cos(2t) + C_2 e^{0t}\operatorname{sen}(2t)$

$y(t) = C_1\cos(2t) + C_2\operatorname{sen}(2t)$; si $y(0) = 0$ (1)

$y'(t) = -2C_1\operatorname{sen}(2t) + 2C_2\cos(2t)$, si $y'(0) = -10$ (2)

$0 = C_1\cos(2(0)) + C_2\operatorname{sen}(2(0)) \rightarrow C_1 = 0$ (1)

(2) $-10 = -2(0)\operatorname{sen}(2(0)) + 2C_2\cos(2(0)) \rightarrow \frac{-10}{2} = C_2 \rightarrow C_2 = -5$

$\therefore y(t) = (0)\cos(2t) - 5\operatorname{sen}(2t)$

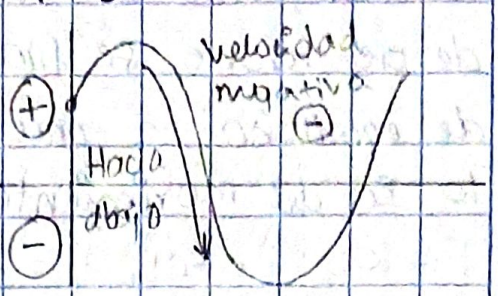
$y(t) = -5\operatorname{sen}(2t) \rightarrow$ Ec. del movimiento

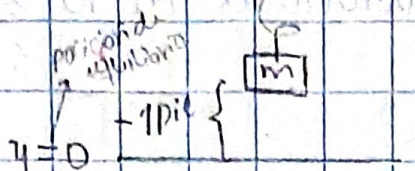
5) Un contrapeso de 32 lb estira 2 pies a un resorte...

$F = kS$. $|A| = ¿?$, $T = ¿?$

$S = 2$ pies. $y(0) = -1$ pie

$y'(0) = -2$ pies/s. $f = ¿?$





$$k = \frac{F}{S} = \frac{32\text{ lb}}{2\text{ pies}} = 16\frac{\text{lb}}{\text{pies}} \qquad \frac{\text{lb}}{\frac{\text{pie}}{s^2}} = \frac{\text{lb}\,s^2}{\text{pie}}$$

$$m = \frac{W}{g} = \frac{32\text{ lb}}{32\text{ pie/s}^2} = 1\text{ slug}$$

ω → rapidez angular

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\frac{\text{lb}}{\text{pie}}}{1\text{ slug}}} = \sqrt{16} = 4 \rightarrow \omega = 4$$

Período → $T = ¿?$ → $T = \frac{2\pi}{\omega} = \frac{2\pi}{4}s = \frac{\pi}{2}s$ → Período del movimiento.

Encontrando Amplitud:

Ec. auxiliar:

$$m\frac{d^2y}{dt^2} + ky = 0 \rightarrow y'' + 16y = 0 \rightarrow M^2 + 16 = 0$$

$$\sqrt{M^2} = \sqrt{-16} \rightarrow M = \pm 4i,\ \alpha = 0,\ \beta = 4$$

$$y(t) = C_1 e^{0t}\cos(4t) + C_2 e^{0t}\operatorname{sen}(4t) = C_1\cos(4t) + C_2\operatorname{sen}(4t)$$

Si $y(0) = -1 \rightarrow -1 = C_1\cos(4(0)) + C_2\operatorname{sen}(4(0)) \rightarrow C_1 = -1$

$y'(t) = -4C_1\operatorname{sen}(4t) + 4C_2\cos(4t)$, Si $y'(0) = -2$

$$-2 = -4(-1)\operatorname{sen}(4(0)) + 4C_2\cos(4(0))$$

$$-2 = 4C_2 \rightarrow \frac{-2}{4} = C_2 \rightarrow C_2 = -\frac{1}{2}$$

$y(t) = -\cos(4t) - \frac{1}{2}\operatorname{sen}(4t)$; Si $|A| = \sqrt{C_1^2 + C_2^2}$

$\therefore |A| = \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \rightarrow |A| = \frac{\sqrt{5}}{2}$

Vibraciones hasta los 4π segundos.

$\omega = 2\pi f \rightarrow f = \frac{\omega}{2\pi} = \frac{4}{2\pi} = \frac{2}{\pi}(4\pi) = 8$ vibraciones

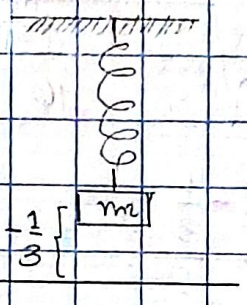
6) Una masa pesa 10 lb y estira $\frac{1}{4}$ pie un resorte. Se quita esa...

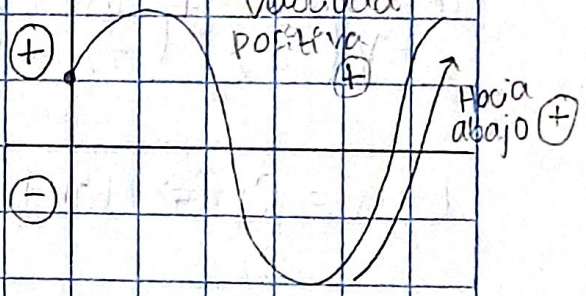
$m_1 = 10$ lb, $s_1 = \frac{1}{4}$ pie.

$m_2 = 1.6$ slugs. $y(0) = -\frac{1}{3}$ pie

$x(t) = A\operatorname{Sen}(\omega t + \phi)$

$y'(0) = \frac{5}{4}\frac{\text{pie}}{\text{s}}$; $k = ?$





$k = \frac{10\text{ lb}}{\frac{1}{4}\text{ pie}} = \frac{40\text{ lb}}{\text{pie}}$ $\quad m\frac{d^2y}{dt^2} + ky = 0$; $m_2 = 1.6 = \frac{8}{5}$ slugs

$\frac{8}{5}y'' + 40y = 0 \rightarrow$ Ec. auxiliar $\rightarrow \left[\frac{5}{8}\right]\left[\frac{8}{5}M^2 + 40\right] = \left[\frac{5}{8}\right][0]$

$M^2 + 25 = 0 \rightarrow \sqrt{M^2} = \sqrt{-25} \rightarrow M = \sqrt{-25(-1)} \rightarrow M = \pm 5i$

$\alpha = 0$, $\beta = 5$, $y(t) = C_1 e^{\alpha t}\cos(5t) + C_2 e^{\alpha t}\operatorname{sen}(5t)$

$y(t) = C_1 e^{0(t)}\cos(5t) + C_2 e^{(0)(t)}\operatorname{sen}(5t) = C_1\cos(5t) + C_2\operatorname{sen}(5t)$; Si $y(0) = -\frac{1}{3}$

$y'(t) = -5C_1\operatorname{sen}(5t) + 5C_2\cos(5t)$; Si $y'(0) = \frac{5}{4}$

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$$-\frac{1}{3} = C_1 \cos(5(0)) + \cancel{C_2 \operatorname{sen}(5(0))}^{\to 0} \longrightarrow \boxed{C_1 = -\frac{1}{3}}$$

$$\frac{5}{4} = \cancel{-5\left(-\frac{1}{3}\right)\operatorname{sen}(5(0))}^{\to 0} + 5C_2 \cancel{\cos(5(0))}^{\to 1} \longrightarrow \boxed{C_2 = \frac{\cancel{5}}{\cancel{20}} = \frac{1}{4}}$$

$$y(t) = -\frac{1}{3}\cos(5t) + \frac{1}{4}\operatorname{sen}(5t)$$

Pasando a forma $x(t) = A\operatorname{sen}(\omega t + \phi)$

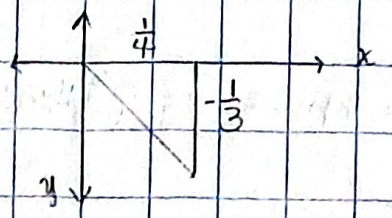
$|A| = ¿? \longrightarrow |A| = \sqrt{C_1^2 + C_2^2}$

$$|A| = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{\frac{1}{9} + \frac{1}{16}} = \sqrt{\frac{16+9}{144}} = \sqrt{\frac{25}{144}} = \frac{5}{12}$$

$$\tan\phi = \frac{C_1}{C_2} \Rightarrow \tan\phi = \frac{-\frac{1}{3}}{\frac{1}{4}} = -\frac{4}{3} \longrightarrow \phi = \tan^{-1}\left(-\frac{4}{3}\right)$$

$\phi = -0.9273$; $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{40}{\frac{8}{5}}} = 5$

Si $\tan\phi = \frac{-\frac{1}{3}}{\frac{1}{4}}$ ∴ →



∴ $x(t) = \frac{5}{12}\operatorname{sen}(5t - 0.9273)$

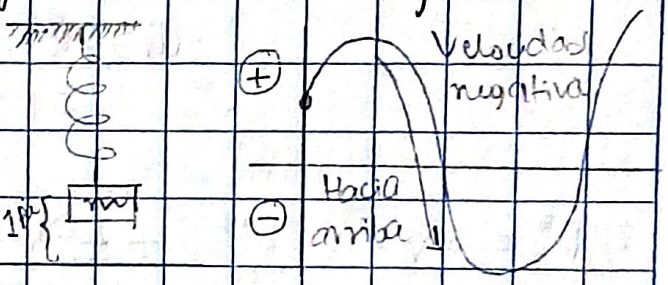
$$|A| = \frac{\frac{5}{12}}{2} = \frac{5}{24}$$

$\frac{5}{24} = \frac{5}{12}\operatorname{sen}(5t - 0.9273) \rightarrow \frac{1}{2} = \operatorname{sen}(5t - 0.9273)$

$\operatorname{Sen}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\quad$ pero $\quad \frac{\pi}{6} = 5t - 0.9273 \rightarrow \frac{\pi}{30} + \frac{0.9273}{5} = t$

$t = 0.2902$ segundos ; si $\frac{\Delta}{2}$ sucede en más de 1 ocasión, se hace $\rightarrow T = \frac{2\pi}{w} = \frac{2\pi}{5} = \frac{2\pi}{5}$ cada $t =$

7) Se cuelga una masa de 1 slug de un resorte cuya constante es 9 lb/pie...



$m = 1$ slug. $k = 9 \frac{lb}{pie}$

$y(0) = -1$ pie, $y'(0) = -\sqrt{3}\frac{pie}{s}$

$m\frac{d^2y}{dt^2} + ky = 0$

$1y'' + 9y = 0$, Ec. aux. $M^2 + 9 = 0$

$\sqrt{M^2} = \sqrt{-9} \rightarrow M_1 = \pm 3i$, $\alpha = 0$, $\beta = 3$

$y(t) = C_1\cos(3t) + C_2\operatorname{sen}(3t)$; si $y(0) = -1$

$y'(t) = -3C_1\operatorname{sen}(3t) + 3C_2\cos(3t)$, si $y'(0) = -\sqrt{3}$

$-1 = C_1\cancel{\cos(3(0))}^{1} + \cancel{C_2\operatorname{sen}(3(0))}^{0} \rightarrow C_1 = -1$

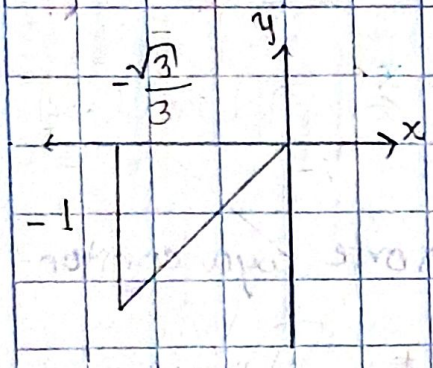
$-\sqrt{3} = \cancel{-3(-1)\operatorname{sen}(3(0))}^{0} + 3C_2\cancel{\cos(3(0))}^{1} \rightarrow C_2 = -\frac{\sqrt{3}}{3}$

$y(t) = -\cos(3t) - \frac{\sqrt{3}}{3}\operatorname{sen}(3t)$

Parando a forma $y(t) = A\,sen(\omega t + \phi)$

$$|A| = \sqrt{C_1^2 + C_2^2} = \sqrt{(-1)^2 + \left(-\frac{\sqrt{3}}{3}\right)^2} = \sqrt{1 + \frac{3}{9}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\phi = \frac{C_1}{C_2} = \frac{-1}{-\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{3} = \sqrt{3}$$



$$\phi = \tan^{-1}(\sqrt{3}) = 1.0472\left(\frac{180}{\pi}\right) = 60$$

$$\phi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}, \quad \omega = \sqrt{\frac{9}{1}} = \sqrt{9} = 3$$

$$y(t) = \frac{2\sqrt{3}}{3}\,sen\left(3t + \frac{4\pi}{3}\right)$$

$$y'(t) = 2\sqrt{3}\cos\left(3t + \frac{4\pi}{3}\right) \rightarrow \text{Si}^{\circ}\ y'(t) = 3\ \frac{pie}{s}$$

$$3 = 2\sqrt{3}\cos\left(3t + \frac{4\pi}{3}\right) \rightarrow \frac{3}{2\sqrt{3}} = \cos\left(3t + \frac{4\pi}{3}\right)$$

$$\left\{\frac{3}{2\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}\right.$$

$$\cos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \rightarrow \frac{\pi}{6} = 3t + \frac{4\pi}{3} \rightarrow 3t = \frac{\pi}{6} - \frac{4\pi}{3}$$

$$3t = -\frac{7\pi}{6} \rightarrow t = -\frac{7\pi}{18}; \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{3} = \frac{2n\pi}{3}$$

$$0' \quad -\frac{\pi}{6} = 3t + \frac{4\pi}{3} \rightarrow 3t = -\frac{\pi}{6} + \frac{4\pi}{3} \Rightarrow t = -\frac{\pi}{2}$$

Arimany.

∴ $t_1 = \left(\frac{-\pi}{2} + \frac{2n\pi}{3}\right)s$; $t_2 = \left(\frac{-7\pi}{18} + \frac{2n\pi}{3}\right)s$ $\Big\}$ $n = 1, 2, 3 \ldots$

8) Para la Ec. del movimiento $y = C_1\cos(\omega t) + C_2 \operatorname{sen}(\omega t)$ ...
se obtiene $y(t) = y_0\cos(\omega t) + \frac{v_0}{\omega}\operatorname{sen}(\omega t)$

$y(0) = y_0$, $y'(0) = v_0$

$y_0 = C_1\cos(\omega(0)) + C_2 \operatorname{sen}(\omega(0)) \rightarrow C_1 = y_0$

$y'(t) = -\omega C_1 \operatorname{sen}(\omega t) + \omega C_2\cos(\omega t)$

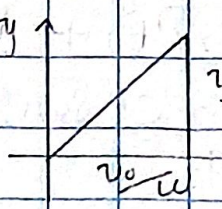
$v_0 = -\omega(y_0)\operatorname{sen}(\omega(0)) + \omega C_2\cos(\omega(0)) \rightarrow C_2 = \frac{v_0}{\omega}$

∴ $y(t) = y_0\cos(\omega t) + \frac{v_0}{\omega}\operatorname{sen}(\omega t)$

Haciendo en forma alternativa $y = A\operatorname{sen}(\omega t + \phi)$

$|A| = \sqrt{C_1^2 + C_2^2} = \sqrt{(y_0)^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{y_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{\frac{y_0^2\omega^2 + v_0^2}{\omega^2}}$

$\tan\phi = \frac{C_1}{C_2} = \frac{y_0}{\frac{v_0}{\omega}} = \frac{y_0\omega}{v_0}$



$\phi = \tan^{-1}\left(\frac{y_0\omega}{v_0}\right)$

$y = \frac{\sqrt{y_0^2\omega^2 + v_0^2}}{\omega}\operatorname{sen}\left(\omega t + \tan^{-1}\left(\frac{y_0\omega}{v_0}\right)\right)$ → Ec. alternativa

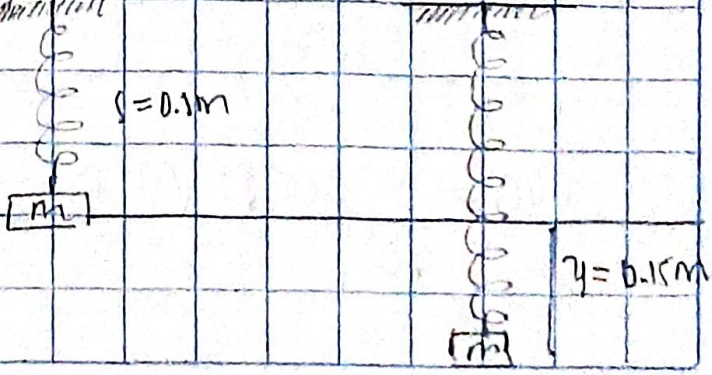
9) Si una bola de hierro de peso $W = 89\,N$ estirara $10\,cm$ un ...

$W = 89N$, $S_1 = 10cm = 0.1m$

$f = ¿?$; $S_2 = 0.15m$

$k = \frac{89N}{0.1m} = 890\frac{N}{m}$

$m = \frac{89N}{9.8\frac{m}{s^2}} = 9.0816 \approx 9.082\,kg$



Ardmany.

$m\frac{d^2y}{dt^2} + Ky = 0 \rightarrow 9.037y'' + 890y = 0 \rightarrow y'' + 98y = 0$

Ec. auxiliar $\rightarrow M^2 + 98 = 0 \rightarrow \sqrt{M^2} = \sqrt{-98} \rightarrow M = \sqrt{+98(-1)}$

$M = \sqrt{98}\sqrt{-1} \rightarrow M = \sqrt{7^2 \cdot 2}\, i \rightarrow M = \pm 7\sqrt{2}\, i$

$\alpha = 0$, $\beta = 7\sqrt{2}$:

$y(t) = C_1 \cos(7\sqrt{2}\,t) + C_2 \operatorname{sen}(7\sqrt{2}\,t)$

Si $\omega = 7\sqrt{2}$ ∴ $\omega = 2\pi f \rightarrow f = \frac{\omega}{2\pi} = \frac{7\sqrt{2}}{2\pi}$

$f = \frac{1.5756}{s} \left| \frac{60\ s}{1\ min} \right| = 94.54\ min \cdot ciclos \approx 95\ ciclos\ por\ min.$

¿Cuál será su mov. si el piso se jala hacia abajo otros 15 cm?

$y(0) = 0.15$, $y'(0) = 0$ $\qquad -0.15 = \frac{3}{20}$

$y'(t) = -7\sqrt{2}\, C_1 \operatorname{sen}(7\sqrt{2}\,t) + 7\sqrt{2}\, C_2 \cos(7\sqrt{2}\,t)$

$\frac{3}{20} = C_1 \cos(7\sqrt{2}\,(0)) + C_2 \operatorname{sen}(7\sqrt{2}\,(0)) \rightarrow C_1 = \frac{3}{20}$

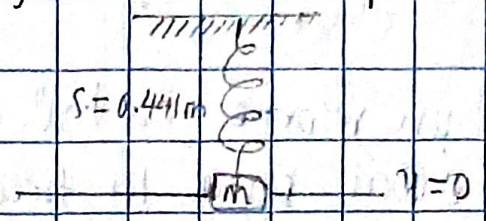
$0 = \cancel{-7\sqrt{2}\, C_1 \operatorname{sen}(7\sqrt{2}\,(0))}^{\,0} + 7\sqrt{2}\, C_2 \cancel{\cos(7\sqrt{2}\,(0))}^{\,1}$

$C_2 = 0$

∴ $y(t) = \frac{3}{20}\cos(7\sqrt{2}\,t) + \cancel{0\,[\operatorname{sen}(7\sqrt{2}\,t)]}^{\,0}$

$y(t) = \frac{3}{20}\cos(7\sqrt{2}\,t) \rightarrow$ Ec. del movimiento

10) Un cuerpo pesa 67.77 N y estira un resorte una longitud de 0.441 m llevándolo a la posición de equilibrio, ¿cuál es el valor de k y cuántos ciclos por minuto ejecutará el cuerpo cuando se le deje oscilar?



$W = 67.77\ N$, $S = 0.441\ m$

$k = ¿?$; $f = ¿? \rightarrow f = \frac{W}{2\pi}$

$k = \frac{67.77\ N}{0.441\ m} = 153.67 \approx 154\ \frac{N}{m}$

$m = \frac{67.77\ kg \cdot m/s^2}{9.8\ \frac{m}{s^2}} = 6.92\ kg$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{153.67}{6.92}} = 4.71$

$f = \frac{4.71}{2\pi} = 0.7496\ s^{-1}$

$f = \frac{0.7496\ ciclos}{s} \left| \frac{60\ s}{1\ min} \right| = 44.976 \approx 45$ ciclos por minuto

11) ¿cómo cambia la frecuencia ω si:

i) se duplica su masa

$\omega = \sqrt{\frac{k}{m}}$ si se duplica su masa $\rightarrow \omega = \sqrt{\frac{k}{2m}} \rightarrow \omega = \sqrt{\frac{1}{2}}\sqrt{\frac{k}{m}} = \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}}$

$\sqrt{2}\,\omega = \sqrt{\frac{k}{m}} \rightarrow$ cambia en factor de $\sqrt{2}$

ii) se usa un resorte más rígido $\omega = \sqrt{\frac{nk}{m}} = \sqrt{n}\sqrt{\frac{k}{m}}$

$\frac{\omega}{\sqrt{n}} = \sqrt{\frac{k}{m}}$; cambia en un factor $\frac{1}{\sqrt{n}}$, donde n puede tomar cualquier valor (+), el cual se multiplicará con la cte del resorte, provocando un valor más grande para dicha cte. llegando a concluir que el resorte será más grueso.

iii) Si se modifican las condiciones iniciales.
NO se modificará el valor de ω, ya sea si se modifica la constante k y la masa m.

12) Si un resorte es tal que un peso de 40 N lo estiraría 4cm, ¿Cuál sería la frecuencia de oscilación correspondiente? ¿El período?.

$W = 40\ N$, $S = 4cm = 0.04m$; $f = ¿?$, $T = ¿?$

$k = \frac{40\ N}{0.04\ m} = 1{,}000\ \frac{N}{m}$; $m = \frac{40N}{9.8\ \frac{m}{s^2}} = 4.0816\ kg$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{4.0816}} = 15.65$; $f = \frac{\omega}{2\pi} = \frac{15.65}{2\pi} = 2.489\ Hz$

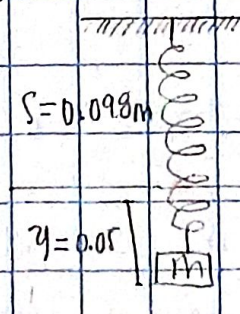
$T = \frac{1}{2.489} = 0.4018\ s$

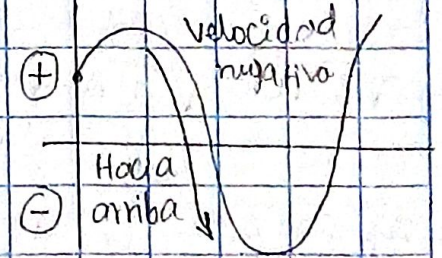
13) Un peso de 20.0 N estira 9.8 cm un resorte, y el peso sujeto al resorte se jala hacia abajo 5cm de la posición de reposo...

$W = 20\ N$, $S = 0.098\ m$

$y(0) = 0.05\ m$, $y'(0) = -0.3\ \frac{m}{s}$

$k = \frac{20\ N}{0.098m} = 204.083\ \frac{N}{m}$





$m = \frac{2.0N}{9.8\ \frac{m}{s^2}} = 2.041\ kg$

$m\frac{d^2y}{dt^2} + ky = 0$

$2.041y'' + 204.083 = 0 \rightarrow \left[\frac{1}{2.041}\right]\left[2.041\,y'' + 204.083\right] = \left[\frac{1}{2.041}\right][0]$

Ec. auxiliar $\rightarrow M^2 + 100 = 0 \rightarrow \sqrt{M^2} = \sqrt{-100} \rightarrow M = \pm 10i$

$\alpha = 0 \wedge \beta = 10$; $y(t) = C_1\cos(10t) + C_2\operatorname{sen}(10t)$; si $y(0) = \frac{1}{20}$

$y'(t) = -10C_1\operatorname{sen}(10t) + 10C_2\cos(10t)$; si $y'(0) = -\frac{3}{10}$

$\frac{1}{20} = C_1\cos(10(0)) + C_2\operatorname{sen}(10(0)) \rightarrow C_1 = \frac{1}{20}$

$-\frac{3}{10} = -10\left(\frac{1}{20}\right)\operatorname{sen}(10(0)) + 10C_2\cos(10(0)) \rightarrow C_2 = -\frac{3}{100}$

$y(t) = \frac{1}{20}\cos(10t) - \frac{3}{10}\operatorname{sen}(10t)$

14) Un objeto de masa disconocida oscila en el extremo...

$m_1 = ?$, $T_1 = 5s$, $W_1 = 2lb$, $T_2 = 7s$, $m_2 = \frac{2lb}{32ft/s^2} = \frac{1}{16}slug$

$\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{5} \rightarrow \omega = \sqrt{\frac{k_1}{m_1}} \rightarrow \left(\frac{2\pi}{5}\right)^2 = \left(\sqrt{\frac{k_1}{m_1}}\right)^2 \rightarrow \frac{4\pi^2}{25} = \frac{k_1}{m_1}$

$k_1 = \frac{4\pi^2(m_1)}{25}$; $\omega_2 = \frac{2\pi}{T_2} = \frac{2\pi}{7} \rightarrow \left(\frac{2\pi}{7}\right)^2 = \left(\sqrt{\frac{k_2}{m_1+m_2}}\right)^2$

$\frac{4\pi^2}{49} = \frac{k_2}{m_1+2} \rightarrow k_2 = \frac{4\pi^2 m_1 + 8\pi^2}{49}$; si $k_1 = k_2$

$$\frac{4\pi^2 m_1}{25} = \frac{(m_1+2)4\pi^2}{49} \rightarrow \frac{m_1}{25} = \frac{m_1+2}{49}$$

$$49m_1 = (m_1+2)25 \rightarrow 49m_1 - 25m_1 = 50$$

$$24m_1 = 50 \rightarrow m_2 = \frac{50}{24} = \frac{25}{12} = 2.0833\ lb = 0.0651\ slug$$

$$\frac{lb}{32\ pie/s^2}$$

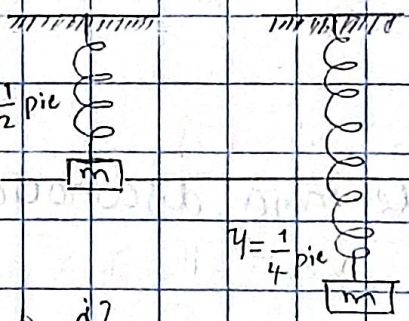
$$m_2 = 0.0651\ slug$$

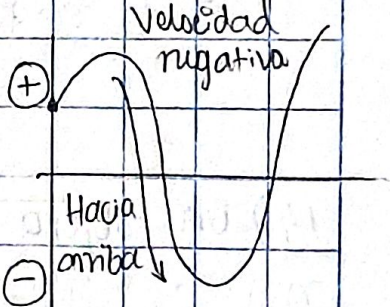
18) Un objeto de 2 libras de peso tira un resorte 6 pulgadas. La masa es tirada hacia abajo otras 3 pulgadas adicionales y después se pone en movimiento. Encuentre el problema con valor inicial del movimiento suponiendo que no hay ninguna amortiguación, y determine el período, la amplitud y el desplazamiento de fase.

$W = 2\ lb$, $y'(0) = -1\ pie/s$

$$S = 6\ pulgadas \left|\frac{1\ pie}{12\ pulg.}\right| = \frac{1}{2}\ pie$$

$$y(0) = 3\ pulg. \left|\frac{1\ pie}{12\ pulg.}\right| = \frac{1}{4}\ pie$$





; $T = |A| = \phi = \dot{c}?$

$$m = \dot{c}? \rightarrow m = \frac{2\ lb}{32\ \frac{pie}{s^2}} = \frac{1}{16}\ slug. \qquad k = \frac{F}{S} = \frac{2\ lb}{\frac{1}{2}\ pie} = 4\ \frac{lb}{pie}$$

$$m\frac{d^2y}{dt^2} + ky = 0 \rightarrow \frac{1}{16}y'' + 4y = 0 \rightarrow \left[\frac{1}{16}y'' + 4y\right]16 = 0\left[16\right]$$

Ec. auxiliar → $M^2 + 64 = 0 \rightarrow \sqrt{M^2} = \sqrt{-64} \rightarrow M = \sqrt{-64(+1)}$

$M = \sqrt{-64}\sqrt{-1} \rightarrow M = \pm 8i$; $\alpha = 0 \wedge \beta = 8$

$y(t) = C_1 e^{\alpha t}\cos(\omega t) + C_2 e^{\alpha t}\sin(\omega t)$

$y(t) = C_1 e^{0t}\cos(8t) + C_2 e^{0t}\sin(8t) = C_1\cos(8t) + C_2\sin(8t)$

$y'(t) = -8C_1\sin(8t) + 8C_2\cos(8t)$

Si $y(0) = \frac{1}{4}$, lo

$\frac{1}{4} = C_1\cos(8(0)) + C_2\sin(8(0)) \rightarrow C_1 = \frac{1}{4}$

Si $y'(0) = -1$ lo

$-1 = -8\left(\frac{1}{4}\right)\sin(8(0)) + 8C_2\cos(8(0)) \rightarrow C_2 = -\frac{1}{8}$

$y(t) = \frac{1}{4}\cos(8t) - \frac{1}{8}\sin(8t)$ → Ec. del movimiento

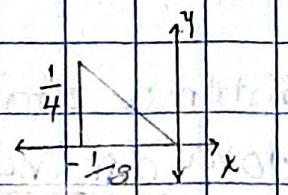
$y(t) = A\sin(8t + \phi)$ → Ec. alternativa.

* Encontrando la Amplitud |A|:

$|A| = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(-\frac{1}{8}\right)^2} = \sqrt{\frac{1}{16} + \frac{1}{64}} = \sqrt{\frac{5}{64}} = \frac{\sqrt{5}}{8}$

Para ϕ

$\tan\phi = \frac{C_1}{C_2} = \frac{\frac{1}{4}}{-\frac{1}{8}} = -2$



$\phi = \tan^{-1}(-2) = -1.1071$

$-1.1071 + \pi = 2.034$

$y(t) = \frac{\sqrt{5}}{8}\sin(8t + 2.034)$

Periodo:

$T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}\ s$

Guía Semana 11: Movimiento Libre Amortiguado.

$$\frac{d^2y}{dt^2} + \frac{\beta}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \quad ; \quad \text{Si } 2\lambda = \frac{\beta}{m} \text{ , } \omega^2 = \frac{k}{m}$$

$$\therefore \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 = 0. \qquad \text{Ec. auxiliar: } m^2 + 2\lambda m + \omega^2 = 0$$

$$m = \frac{-2\lambda \pm \sqrt{(2\lambda)^2 + 4(1)(\omega^2)}}{2(1)} \rightarrow m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

a) Si $\lambda^2 - \omega^2 > 0$, el sistema está sobreamortiguado (no oscila), por lo que β (const. de amortiguamiento) es grande comparado con k (const. del resorte). $\beta > k$

$$\left.\begin{aligned} m_1 &= -\lambda - \sqrt{\lambda^2 - \omega^2} \\ m_2 &= -\lambda + \sqrt{\lambda^2 - \omega^2} \end{aligned}\right\} \quad y(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

b) Si $\lambda^2 - \omega^2 = 0$, el sistema está críticamente amortiguado. Si la fuerza amortiguadora disminuye ligeramente el s. pudo oscilar.

$$m_1 = m_2 = -\lambda \quad ; \quad y(t) = C_1 e^{m_1 t} + C_2 t e^{m_1 t}$$

c) Si $\lambda^2 - \omega^2 < 0$, se dice que el sistema está subamortiguado porque el coeficiente de amortiguamiento es pequeño en comparación con la constante del resorte (k). El sistema sí oscila

$$m_1 = -\lambda - \sqrt{\lambda^2 - \omega^2} \quad \left.\right\} \quad \alpha = -\lambda \;,\; r = \sqrt{\omega^2 - \lambda^2}$$
$$m_2 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$

$$y(t) = C_1 e^{\alpha t}\cos(rt) + C_2 e^{\alpha t}\operatorname{sen}(rt)$$

• Movimiento forzado. Si $f(t) \neq 0$, es decir, que sobre el sistema actúa una fuerza externa, entonces se dice que el movimiento es forzado.

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = f(t)$$

Ejercicios

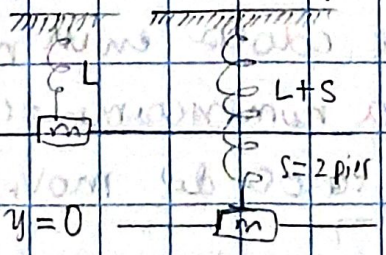
1) Un cuerpo de 8 lb de peso estira 2 pies un resorte. Si una fuerza de amortiguamiento numéricamente igual a dos veces la velocidad instantánea actúa sobre el contrapeso. Deduzca la ec. del movi. si la masa se suelta de la posición de equili. con una velocidad hacia arriba de 3 pies por segundo.

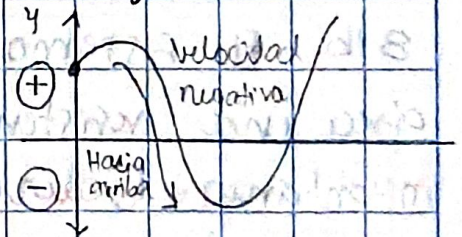
$W = 8\,lb$, $S = 2\,pies$, $\beta = 2$

$y(0) = 0$, $y'(0) = -3\ \frac{pie}{s}$

$$k = \frac{F}{S} = \frac{8\,lb}{2\,pies} = 4\ \frac{lb}{pie}$$

$$m = \frac{W}{g} = \frac{8\,lb}{32\ \frac{pie}{s^2}} = \frac{1}{4}\ slug \;;\; m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$





$$\frac{1}{4}y'' + 2y' + 4y = 0 \rightarrow 4\left[\frac{1}{4}y'' + 2y' + 4y\right] = 0\left[4\right]$$

$y'' + 8y' + 16y = 0$ → Ec. auxiliar $M^2 + 8M + 16 = 0$

$\sqrt{(8)^2 - 4(1)(16)} \rightarrow \sqrt{64 - 64} \rightarrow 0 = 0$; El sistema está críticamente amortiguado.

$M^2 + 8M + 16 = 0 \rightarrow (M+4)(M+4) = 0$

$M_1 = M_2 = -4$, $y(t) = C_1 e^{M_1 t} + C_2 t e^{M_1 t}$

$y(t) = C_1 e^{-4t} + C_2 t e^{-4t}$; si $y(0) = 0$

$y'(t) = -4C_1 e^{-4t} + C_2 e^{-4t} - 4C_2 t e^{-4t}$, si $y'(0) = -3$

Si $y(0) = 0$

$0 = C_1 e^{-4(0)} + C_2 (0) e^{-4(0)} \rightarrow C_1 = 0$

Si $y'(0) = -3$

$-3 = -4(0) e^{-4(0)} + C_2 e^{-4(0)} - 4C_2 (0) e^{-4(0)} \rightarrow C_2 = -3$

$y(t) = (0) e^{-4t} + (-3) t e^{-4t} \rightarrow y(t) = -3t e^{-4t}$ Ec. del movimiento

2) Un resorte de 4 pies alcanza 8 pies al colgarle una pisa de 8 lb. El sistema se coloca en un medio a través del cual se ofrece una resistencia numéricamente igual a $\sqrt{2}$ veces su vel. instantánea. Deduzca la ec. del mov. ...

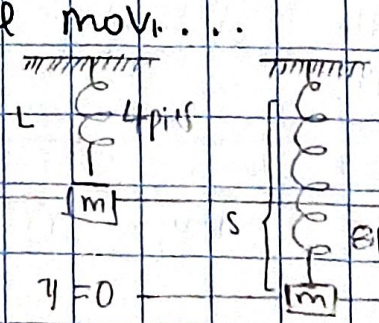
$W = 8$ lb, $S = 8$ pies, $B = \sqrt{2}$

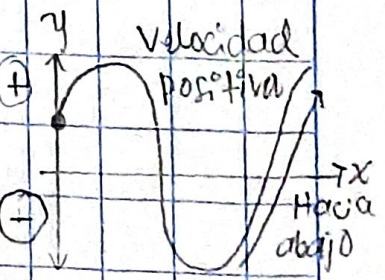
$y(0) = 0$, $y'(0) = 5 \frac{pie}{s}$, $t = ¿?$

$m = \frac{W}{g} = \frac{8\,lb}{32 \frac{pie}{s^2}} = \frac{1}{4}$ Slug.

$S = (8 - 4)$ pies $= 4$ pies

$k = \frac{F}{S} = \frac{8\,lb}{4\,pies} = 2 \frac{lb}{pie}$; $m\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = 0$





1/4 $y'' + \sqrt{2}\,y' + 2y = 0 \rightarrow 4\left[\frac{1}{4}y'' + \sqrt{2}\,y' + y\right] = 0$ [4]

Ec. auxiliar $\rightarrow m^2 + 4\sqrt{2}\,m + 8 = 0$, $\sqrt{(4\sqrt{2})^2 - 4(1)(8)}$

$\sqrt{32 - 32} \rightarrow \sqrt{0} \rightarrow 0 = 0$; El sistema está críticamente amortiguado.

$m = \frac{-4\sqrt{2} \pm \sqrt{0}}{2(1)} = \frac{-4\sqrt{2}}{2} \pm -2\sqrt{2}$; $m_1 = m_2 = -2\sqrt{2}$

$y(t) = C_1 e^{-2\sqrt{2}t} + C_2 t e^{2\sqrt{2}t}$; si $y(0) = 0$

$y'(t) = -2\sqrt{2}\,C_1 e^{-2\sqrt{2}t} + C_2 e^{-2\sqrt{2}t} - 2\sqrt{2}\,C_2 t e^{-2\sqrt{2}t}$; $y'(0) = 5$

si $y(0) = 0$

$0 = C_1 e^{-2\sqrt{2}(0)} + C_2(0)e^{-2\sqrt{2}(0)} \rightarrow C_1 = 0$

si $y'(0) = 5$

$5 = -2\sqrt{2}(0)e^{-2\sqrt{2}(0)} + C_2 e^{-2\sqrt{2}(0)} - 2\sqrt{2}\,C_2(0)e^{-2\sqrt{2}(0)}$; $\rightarrow C_2 = 5$

$y(t) = (0)e^{-2\sqrt{2}t} + 5te^{-2\sqrt{2}t} \rightarrow y(t) = 5te^{-2\sqrt{2}t}$ Ec. del movimiento.

Tiempo en el desplazamiento extremo:

$y'(t) = 0 \quad y'(t) = 5e^{-2\sqrt{2}t} - 10\sqrt{2}te^{-2\sqrt{2}t}$

$0 = 5e^{-2\sqrt{2}t} - 10\sqrt{2}te^{-2\sqrt{2}t} \rightarrow 10\sqrt{2}te^{-2\sqrt{2}t} = 5e^{-2\sqrt{2}t}$

$t = \frac{5e^{-2\sqrt{2}t}}{10\sqrt{2}e^{-2\sqrt{2}t}} = \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4} = 0.35$ segundos.

$y\left(\frac{\sqrt{2}}{4}\right) = 5\left(\frac{\sqrt{2}}{4}\right)e^{-2\sqrt{2}\left(\frac{\sqrt{2}}{4}\right)} = \frac{5\sqrt{2}}{4}e^{-1} = 0.65$ pie

R// 0.65 pie es la posición en el instante de 0.35 segundos.

3. Después de unir una pesa de 10 lb a un resorte de 5 pies, éste mide 7 pies. Se quita y se reemplaza con otra de 8 lb y el sistema se coloca en un medio que ofrece una resistencia numéricamente igual a la velocidad instantánea.

a) Deduzca la ec. del mov. si parte la pesa $\frac{1}{2}$ pie debajo de la posición de equilibrio a una velocidad de $1 \frac{pie}{s}$ hacia abajo.

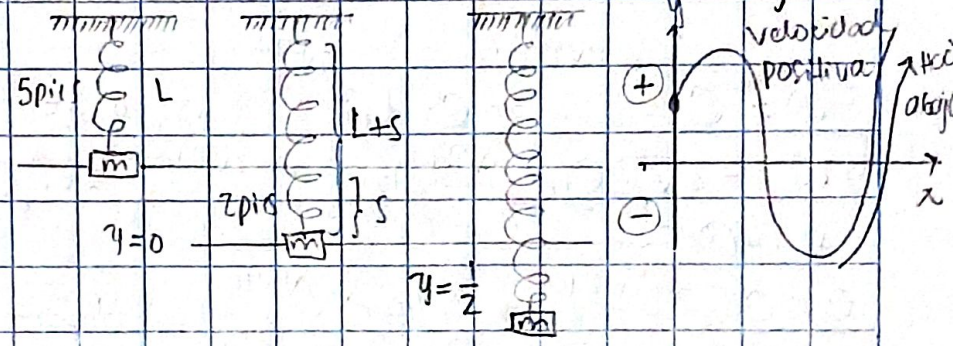
$s = (7-5)\text{ pies} = 2\text{ pies}$,

$g = 1$, $W_1 = 10\text{ lb}$, $W_2 = 8\text{ lb}$

$y(0) = \frac{1}{2}$ pie; $y'(0) = 1 \frac{pie}{s}$

$k = \frac{F_1}{s} = \frac{10\text{ lb}}{2\text{ pies}} = \frac{5\text{ lb}}{\text{pies}}$

$m = \frac{W_2}{g} = \frac{8\text{ lb}}{32 \frac{pie}{s^2}} = \frac{1}{4}$ slug



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0 \rightarrow \frac{1}{4}y'' + 1y' + 5y = 0$$

$$4\left[\frac{1}{4}y'' + y' + 5y\right] = 0\,[4] \rightarrow y'' + 4y' + 20y = 0$$

Ec. auxiliar $\rightarrow m^2 + 4m + 20 = 0$

$\sqrt{(4)^2 - 4(1)(20)} = \sqrt{16-80} = \sqrt{(64)} \rightarrow -64$ 40; El sistema está subamortiguado (sí oscila).

$$m = \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm \sqrt{64}\sqrt{-1}}{2} = -2 \pm 4i \;;\; \alpha = -2,\ \gamma = 4$$

$y(t) = C_1 e^{\alpha t}\cos(\gamma t) + C_2 e^{\alpha t}\operatorname{sen}(\gamma t)$

$y(t) = C_1 e^{-2t}\cos(4t) + C_2 e^{-2t}\operatorname{sen}(4t)$; si $y(0) = \frac{1}{2}$

$y'(t) = -2C_1 e^{-2t}\cos(4t) - 4C_1 e^{-2t}\operatorname{sen}(4t) - 2C_2 e^{-2t}\operatorname{sen}(4t) + 4C_2 e^{-2t}\cos(4t)$; si $y'(0) = 1$

si $y(0) = \frac{1}{2}$

$\frac{1}{2} = C_1 e^{-2(0)}\cos(4(0)) + C_2 e^{-2(0)}\operatorname{sen}(4(0)) \rightarrow \boxed{C_1 = \frac{1}{2}}$

si $y'(0) = 1$

$1 = -2\left(\frac{1}{2}\right)e^{-2(0)}\cos(4(0)) - 4\left(\frac{1}{2}\right)e^{-2(0)}\operatorname{sen}(4(0)) - 2C_2 e^{-2(0)}\operatorname{sen}(4(0)) + 4C_2 e^{-2(0)}\cos(4(0))$

$1 = -1 + 4C_2 \rightarrow 1 + 1 = 4C_2 \rightarrow C_2 = \frac{2}{4} = \frac{1}{2} \rightarrow \boxed{C_2 = \frac{1}{2}}$

$y(t) = e^{-2t}\left[\frac{1}{2}\cos(4t) + \frac{1}{2}\operatorname{sen}(4t)\right]$ → Ec. del movimiento.

b) Exprese la Ec. del movi. en la forma: $x(t) = Ae^{-\lambda t}\operatorname{sen}((\omega^2 - \lambda^2)t + \phi)$

si $2\lambda = 4 \rightarrow \lambda = \frac{4}{2} \rightarrow \lambda = 2 \rightarrow \lambda = -2$

si $\omega^2 = \frac{k}{m} = \frac{5}{\frac{1}{4}} = 20 \rightarrow \omega^2 = 20$

; $\gamma = \sqrt{\omega^2 - \lambda^2} = \sqrt{20 - (-2)^2}$

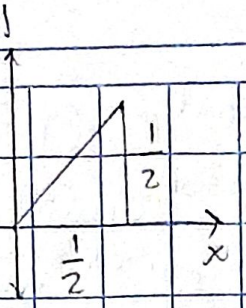
$\gamma = \sqrt{16} = 4$

Encontrando la amplitud $|A|$:

$|A| = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Para ϕ:

$\tan\phi = \frac{C_1}{C_2} = \frac{\frac{1}{2} \to$ c.o $}{\frac{1}{2} \to$ c.a $} = 1$



$\therefore \phi = \tan^{-1}(1) = 0.7854\left(\frac{180}{\pi}\right)$

$\phi = 45°\left(\frac{\pi}{180}\right) \Rightarrow \phi = \frac{\pi}{4}$; $y(t) = \frac{\sqrt{2}}{2} e^{-2t} \operatorname{sen}\left(4t + \frac{\pi}{4}\right)$

c) Calcule los momentos en que el contrapeso pasa por la posición de equilibrio al dirigirse hacia abajo.

Si $y(t) = 0$ ∴

$0 = \frac{\sqrt{2}}{2} e^{-2t} \operatorname{sen}\left(4t + \frac{\pi}{4}\right) \to 0 = \operatorname{sen}\left(4t + \frac{\pi}{4}\right)$

$\operatorname{sen}(0) = 0 \to 0 = 4t + \frac{\pi}{4} \to -\frac{\pi}{16} = t \to$ No puede haber tiempo negativo.

Tomando π:

$\pi = 4t + \frac{\pi}{4} \to \pi - \frac{\pi}{4} = 4t \to \frac{3\pi}{4} = 4t \to t = \frac{3\pi}{16} \to$ Hacia arriba

Tomando 2π:

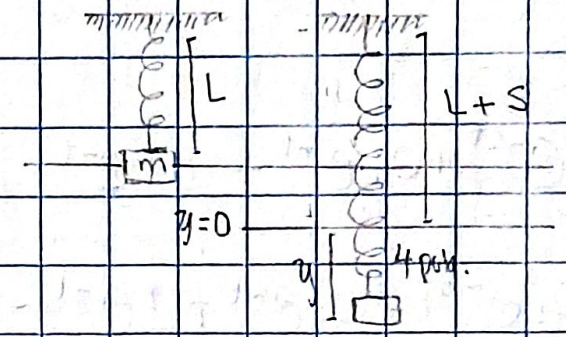
$2\pi = 4t + \frac{\pi}{4} \to 2\pi - \frac{\pi}{4} = 4t \to \frac{7\pi}{16} = t \to$ Hacia abajo por 1era vez.

Tomando 4π:

$4\pi = 4t + \frac{\pi}{4} \to 4\pi - \frac{\pi}{4} = 4t \to \frac{15\pi}{16}$; $\to$ 2da vez hacia arriba.

4) Un peso de 32 lb se sujeta a un resorte de cont. $k = 5 \frac{lb}{pie}$. El peso y el resorte se sumerge en un medio que ofrece una resistencia numéricamente igual a 6 veces la velocidad instantánea. El mov. se inicia en un punto que se encuentra a 4 pulg. abajo de la posici. de equi. y partiendo del reposo. Determine:

a) La ec. del mov. si sobre el peso se aplica una fuerza externa igual a $f(t) = e^{-t}$.



$W = 32\ lb$, $k = 5 \frac{lb}{pie}$ → $y(0) = \frac{1}{3}$ pie

$\beta = 6$, $y(0) = 4$ pulg. $y'(0) = 0$

$m = \frac{W}{g} = \frac{32\ lb}{32 \frac{pie}{s^2}} = 1$ slug.

$y'' + 6y' + 5y = e^{-t}$ $\quad [D - \alpha]^n \quad n-1=0,\ n=1 \ ; \ [D+1]^1$

$(D+1)(D^2+6D+5) = (D+1)[e^{-t}]$

$(D+1)(D^2+6D+5) = [D(e^{-t}) + 1(e^{-t})]$

$(D+1)(D^2+6D+5) = -e^{-t} + e^{-t}$

$(D+1)(D^2+6D+5) = 0$

Ec. auxiliar → $\underbrace{(m+1)}_{\text{para } y_p}\underbrace{(m^2+6m+5)}_{\text{para } y_c} = 0$

Para y_c:

$(m^2+6m+5) = 0 \rightarrow (m+5)(m+1) = 0$; $m_1 = -5 \wedge m_2 = -1$

$y(t) = C_1 e^{-5t} + C_2 e^{-t}$

Para y_p: → $(m+1) = 0 \rightarrow m_3 = -1$

$y_p = Ate^{-t}$

$y'_p = Ae^{-t} - Ate^{-t}$; $y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$y'' = -2Ae^{-t} + Ate^{-t}$

$6y' = 6Ae^{-t} - 6Ate^{-t}$

$5y = 5Ate^{-t}$

$y'' + 6y' + 5y = 4Ae^{-t}$

$4Ae^{-t} = e^{-t} \rightarrow 4A = \frac{e^{-t}}{e^{-t}} \rightarrow A = \frac{1}{4}$

$\therefore y_p = \frac{1}{4} te^{-t}$

$y(t) = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{4} te^{-t}$; si $y(0) = \frac{1}{3}$

$y'(t) = -5c_1 e^{-5t} - c_2 e^{-t} + \frac{1}{4} e^{-t} - \frac{1}{4} te^{-t}$; si $y'(0) = 0$

si $y(0) = \frac{1}{3}$

$\frac{1}{3} = c_1 e^{-5(0)} + c_2 e^{-0} \rightarrow \frac{1}{3} - c_2 = c_1 \rightarrow c_1 = \frac{1}{3} - \frac{17}{48} \rightarrow c_1 = \frac{-1}{48}$

si $y'(0) = 0$

$0 = -5\left(\frac{1}{3} - c_2\right) e^{-5(0)} - c_2 e^{-0} + \frac{1}{4} e^{-0} - \frac{1}{4}(0) e^{-0}$

$0 = -\frac{5}{3} + 5c_2 - c_2 + \frac{1}{4}$

$\frac{5}{3} - \frac{1}{4} = 4c_2 \rightarrow \frac{20 - 3}{12} = 4c_2 \rightarrow \frac{17}{48} = c_2$

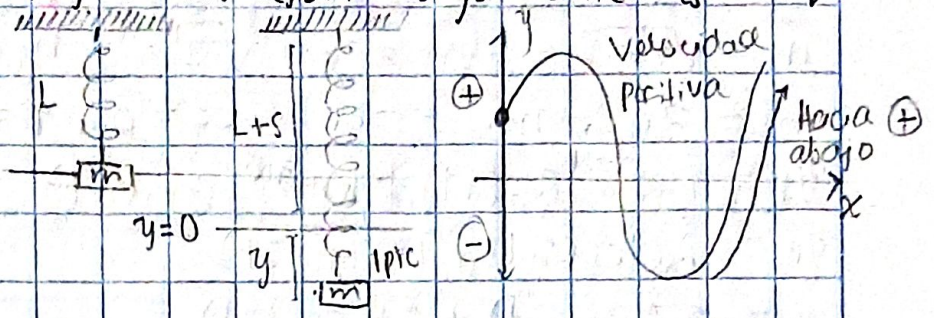
$y(t) = -\frac{1}{48} e^{-5t} + \frac{17}{48} e^{-t} + \frac{1}{4} te^{-t}$ → Ec. del movimiento.

5) Se une una masa de 1 slug a un resorte cuya cnte. es 5 lb/pie. Se suelta la masa...

$m = 1$ slug. $k = 5 \frac{lb}{pie}$;

$y(0) = 1$ pie, $y'(0) = 5 \frac{pie}{s}$.

$\beta = 2$.



a) Deduzca la Ec. del mov. si una fuerza externa dada por $f(t) = 12\cos(2t) + 3\sin(2t)$ actúa sobre la masa.

$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n \quad n-1=0 \rightarrow n=1$; $\alpha = 0$, $\beta = 2$

$[D^2 - 2(0)D + (0^2 + 2^2)]^1 \rightarrow [D^2 + 4]$

$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = f(t) \rightarrow y'' + 2y' + 5y = 12\cos(2t) + 3\sin(2t)$

$(D^2+4)(D^2+2D+5) = [D^2+4][12\cos(2t) + 3\sin(2t)]$

$(D^2+4)(D^2+2D+5) = [D^2(12\cos(2t) + 3\sin(2t)) + 4(12\cos(2t) + 3\sin(2t))]$

$(D^2+4)(D^2+2D+5) = [D(-24\sin(2t) + 6\cos(2t) + 48\cos(2t) + 12\sin(2t)]$

$(D^2+4)(D^2+2D+5) = -48\cos(2t) + 12\sin(2t) + 48\cos(2t) + 12\sin(2t)$

$(D^2+4)(D^2+2D+5) = 0$

Ec. auxiliar $\rightarrow$ $(m^2+4)(m^2+2m+5) = 0$

(m^2+4) para y_p; (m^2+2m+5) para y_c

Para $y_c \rightarrow (m^2+2m+5) = 0$ $\rightarrow$ El sistema está críticamente amortiguado

$m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}\sqrt{-1}}{2} = -1 \pm 2i$

$\alpha = -1$, $r = 2$

$y(t)_c = C_1 e^{\alpha t}\cos(rt) + C_2 e^{\alpha t}\sin(rt)$

$y(t)_c = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$

Para $y_p \rightarrow (m^2+4)=0 \rightarrow m^2=-4 \rightarrow \sqrt{m^2}=\sqrt{-4}$

$m=\sqrt{4(-1)} \rightarrow m=\sqrt{4}\sqrt{-1} \rightarrow m=\pm 2i$

$m_3=2i$, $m_4=-2i$, $\alpha=0$, $r=2$

$y_p=Ae^{\alpha t}\cos(rt)+Be^{\alpha t}\operatorname{sen}(rt)$

$y_p=A\cos(2t)+B\operatorname{sen}(2t)$

$y'_p=-2A\operatorname{sen}(2t)+2B\cos(2t)$

$y''_p=-4A\cos(2t)-4B\operatorname{sen}(2t)$

$y''=+4A\cos(2t)-4B\operatorname{sen}(2t)$

$2y'=-4A\operatorname{sen}(2t)+4B\cos(2t)$

$5y=5A\cos(2t)+5B\operatorname{sen}(2t)$

$y''+2y'+5y=A\cos(2t)-4A\operatorname{sen}(2t)+B\operatorname{sen}(2t)+4B\cos(2t)$

$A\cos(2t)-4A\operatorname{sen}(2t)+B\operatorname{sen}(2t)+4B\cos(2t)=12\cos(2t)+3\operatorname{sen}(2t)$

$(A+4B)\cos(2t)=12\cos(2t)$

$A+4B=\frac{12\cos(2t)}{\cos(2t)} \rightarrow A+4B=12 \rightarrow A=12-4B=12-4(3)$

$\boxed{A=0}$

$(-4A+B)\operatorname{sen}(2t)=3\operatorname{sen}(2t)$

$-4B+16B+B=\frac{3\operatorname{sen}(2t)}{\operatorname{sen}(2t)} \rightarrow 17B=3+4B \rightarrow \boxed{B=3}$

$y_p=(0)\cos(2t)+3\operatorname{sen}(2t) \rightarrow y_p=3\operatorname{sen}(2t)$

$y(t)=C_1e^{-t}\cos(2t)+C_2e^{-t}\operatorname{sen}(2t)+3\operatorname{sen}(2t)$

$y'(t)=-C_1e^{-t}\cos(2t)-2C_1e^{-t}\operatorname{sen}(2t)-C_2e^{-t}\operatorname{sen}(2t)$
$+2C_2e^{-t}\cos(2t)+6\cos(2t)$

$y(0) = 1$

$1 = C_1 e^{-0}\cos(2(0)) + C_2 e^{-0}\operatorname{sen}(2(0)) + 3\operatorname{sen}(2(0)) \rightarrow \boxed{C_1 = 1}$

$y'(0) = 5$

$5 = -(1)e^{-0}\cos(2(0)) - 2(1)e^{-0}\operatorname{sen}(2(0)) - C_2 e^{-0}\operatorname{sen}(2(0)) + 2C_2 e^{-0}\cos(2(0)) + 6\cos(2(0))$

$5 = -1 + 2C_2 + 6 \rightarrow 5 + 1 - 6 = 2C_2 \rightarrow \boxed{C_2 = 0}$

$y(t) = e^{-t}\cos(2t) + (0)e^{-t}\operatorname{sen}(2t) + 3\operatorname{sen}(2t)$

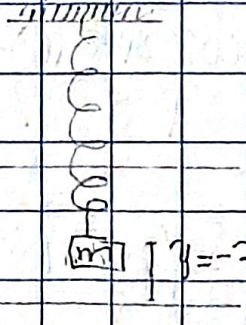
$y(t) = e^{-t}\cos(2t) + 3\operatorname{sen}(2t)$

6) Suponga que una masa de 2kg alarga 5m un resorte. Determinar la ec. del mov. libre amortiguado si la masa ...

$m = 2kg$, $s = 5m$, $\beta = 4$

$y(0) = -2m$, $y'(0) = 0$

$w = m \cdot g \rightarrow w = (2)(9.8) = 19.6$



$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$, $k = \frac{19.6}{5} = 3.92$

$2y'' + 4y' + 3.92y = 0 \rightarrow \frac{1}{2}[2y'' + 4y' + 3.92y] = \frac{1}{2}[0]$

Ec auxiliar $\rightarrow m^2 + 2m + \frac{49}{25} = 0$

$m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(\frac{49}{25})}}{2(1)} = -1 \pm \frac{\sqrt{\frac{96}{25}}\sqrt{-1}}{2} = -1 \pm \frac{\sqrt{4^2 \cdot 6}}{5}\,i$

El sistema esta sub-amortiguado.

$m = -1 \pm \frac{2\sqrt{6}}{5}$; $m_1 = -1 + \frac{2\sqrt{6}}{5}$, $m_2 = -1 - \frac{2\sqrt{6}}{5}$, $\alpha = -1$, $\gamma = -\frac{2\sqrt{6}}{5}$

$$y(t) = C_1 e^{-t}\cos\left(\frac{2\sqrt{6}}{5}t\right) + C_2 e^{-t}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}t\right) \quad ; \text{ si } y(0) = -2$$

$$y'(t) = -C_1 e^{-t}\cos\left(\frac{2\sqrt{6}}{5}t\right) - \frac{2\sqrt{6}}{5} C_1 e^{-t}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}t\right) - C_2 e^{-t}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}t\right)$$

$$+\frac{2\sqrt{6}}{5} C_2 \cos\left(\frac{2\sqrt{6}}{5}t\right) \quad ; \text{ si } y'(0) = 0$$

Si $y(0) = -2$

$$-2 = C_1 e^{-0}\cos\left(\frac{2\sqrt{6}}{5}(0)\right) + C_2 e^{-0}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}(0)\right) \rightarrow \boxed{C_1 = -2}$$

Si $y'(0) = 0$

$$0 = -(-2)e^{-0}\cos\left(\frac{2\sqrt{6}}{5}(0)\right) - \frac{2\sqrt{6}}{5}(-2)e^{-0}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}(0)\right) - C_2 e^{-0}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}(0)\right)$$

$$+\frac{2\sqrt{6}}{5} C_2 \cos\left(\frac{2\sqrt{6}}{5}(0)\right)$$

$$0 = 2 + \frac{2\sqrt{6}}{5} C_2 \rightarrow \frac{-2}{\frac{2\sqrt{6}}{5}} = C_2 \rightarrow \frac{-10}{2\sqrt{6}}\left(\frac{\sqrt{6}}{\sqrt{6}}\right) \Rightarrow \boxed{C_2 = -\frac{5\sqrt{6}}{6}}$$

$$y(t) = -2e^{-t}\cos\left(\frac{2\sqrt{6}}{5}t\right) - \frac{5\sqrt{6}}{6} e^{-t}\operatorname{sen}\left(\frac{2\sqrt{6}}{5}t\right)$$

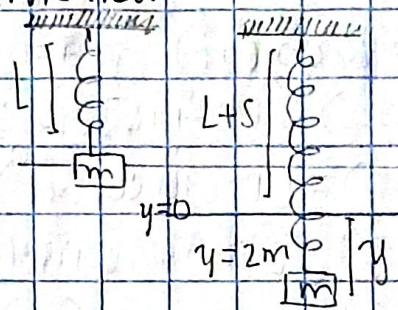
Ec. del movimiento

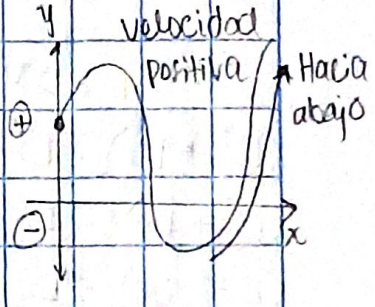
7) Consideremos una masa de 5 kg sujeta a un resorte de const. $k = 20 \frac{N}{m}$. Sobre el sistema está actuando una fuerza amortiguadora de constante $B = 20 \frac{Ns}{m}$. Si la masa se suelta 2m por debajo de su posic. de equi. con una velo. inicial descendente $1 \frac{m}{s}$, determine la ec. del movi. del sistema.

$m = 5\ kg$, $K = 20 \frac{N}{m}$, $y'(0) = 1 \frac{m}{s}$

$B = 20 \frac{Ns}{m}$, $y(0) = 2\ m$

$W = ? = m \cdot g = (5kg)(9.8 \frac{m}{s^2}) = 49\ N$





$$m\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = 0 \rightarrow 5y'' + 20y' + 20y = 0$$

$$\frac{1}{5}\left[5y'' + 20y' + 20y\right] = \frac{1}{5}[0] \rightarrow y'' + 4y' + 4y = 0$$

Ec. auxiliar:

$m^2 + 4m + 4 = 0 \rightarrow \sqrt{(4)^2 - 4(1)(4)} \rightarrow \sqrt{16-16} \Rightarrow \sqrt{0} \rightarrow 0 = 0$

El sistema está críticamente amortiguado

$(m+2)(m+2) = 0 \rightarrow m_1 = m_2 = -2$

$y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$; Si $y(0) = 2$

$y'(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$; Si $y'(0) = 1$

$2 = C_1 e^{-2(0)} + C_2(0)e^{-2(0)} \rightarrow \boxed{C_1 = 2}$

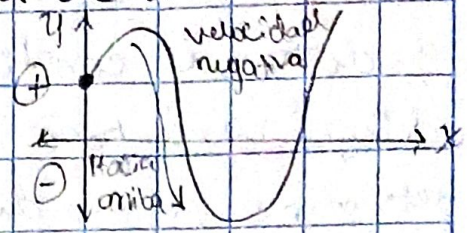
$1 = -2(2)e^{-2(0)} + C_2 e^{-2(0)} - 2C_2(0)e^{-2(0)} \rightarrow 1 + 4 = C_2 \rightarrow \boxed{C_2 = 5}$

$$y(t) = e^{-2t}[2 + 5t]$$

a) Determinar la Ec. del mov. de un sistema masa-resorte formado por una masa de 8 kg suspendida de un resorte...

$m = 8kg$, $k = 32 \frac{N}{m}$, $y(0) = -1 \frac{m}{s}$

$f(t) = 16\cos(4t)$. $y(0) = 0$



$8y'' + 32y = 16\cos(4t)$, $n-1=0 \rightarrow n=1$, $[D^2+16]^1$

$(D^2+16)(8D^2+32) = [D^2+16](16\cos(4t))$

$(D^2+16)(8D^2+32) = [D^2(16\cos(4t)) + 16(16\cos(4t))]$

$(D^2+16)(8D^2+32) = [D(-64\sen(4t)) + 256\cos(4t)]$

$(D^2+16)(8D^2+32) = -256\cos(4t) + 256\cos(4t)$

$(D^2+16)(8D^2+32) = 0$

Ec. auxiliar $\rightarrow$ $(m^2+16)(8m^2+32) = 0$

(m^2+16): Para y_p — ($8m^2+32$): para y_c

Para $y_c \rightarrow (8m^2+32) = 0 \rightarrow \sqrt{m^2} = \sqrt{-4} \rightarrow m = \pm 2$

$m_1 = 2i$, $m_2 = -2i$, $\alpha = 0$, $r = 2$

$y_c = C_1\cos(2t) + C_2\sen(2t)$

Para $y_p \rightarrow (m^2+16) = 0 \rightarrow m^2 = -16 \rightarrow m = \pm 4i$

$m_3 = 4i$, $m_4 = -4i$, $\alpha = 0$, $r = 4$

$y_p = A\cos(4t) + B\sen(4t)$

$y'_p = -4A\sen(4t) + 4B\cos(4t)$; $y''_p = -16A\cos(4t) - 16B\sen(4t)$

$8y'' = -128A\cos(4t) - 128B\sen(4t)$

$32y = 32A\cos(4t) + 32B\sen(4t)$

$8y'' + 32y = -96A\cos(4t) - 96B\sen(4t)$

$-96A\cos(4t) - 96B\sen(4t) = 16\cos(4t)$

$-96A = 16 \rightarrow A = -\frac{1}{6}$

$-96B = 0 \rightarrow B = 0$

$y_p = \frac{-1}{6}\cos(4t) + (0)\operatorname{sen}(4t) \rightarrow y_p = \frac{-1}{6}\cos(4t)$

$y(t) = C_1\cos(2t) + C_2\operatorname{sen}(2t) - \frac{1}{6}\cos(4t)$; Si $y(0) = 0$

$y'(t) = -2C_1\operatorname{sen}(2t) + 2C_2\cos(2t) + \frac{2}{3}\operatorname{sen}(4t)$; Si $y'(0) = -1$

$0 = C_1\cos(2(0)) + C_2\operatorname{sen}(2(0)) - \frac{1}{6}\cos(4(0)) \rightarrow C_1 = \frac{1}{6}$

$-1 = -2\left(\frac{1}{6}\right)\operatorname{sen}(2(0)) + 2C_2\cos(2(0)) + \frac{2}{3}\operatorname{sen}(4(0)) \rightarrow C_2 = -\frac{1}{2}$

$y(t) = \frac{1}{6}\cos(2t) - \frac{1}{2}\operatorname{sen}(2t) - \frac{1}{6}\cos(4t)$

9) A un sistema masa-resorte no amortiguado cuyos parámetros son $m = 1\,kg$, $k = 9\,\frac{N}{m}$ se le aplica una fuerza externa dada por $F(t) = 6\operatorname{sen}(3t)$. Determinar la ec. que describe el movi. del sist. suponiendo que $y(0) = 1\,m$, $y'(0) = 0$.

$y'' + 9y = 6\operatorname{sen}(3t)$, $[D^2 + 9]$

$(D^2+9)(D^2+9) = [D^2(6\operatorname{sen}(3t)) + 9(6\operatorname{sen}(3t))]$

$(D^2+9)(D^2+9) = [D(18\cos(3t) + 54\operatorname{sen}(3t)]$

$(D^2+9)(D^2+9) = -54\operatorname{sen}(3t) + 54\operatorname{sen}(3t)$

$(D^2+9)(D^2+9) = 0$

Ec. auxiliar $\rightarrow$ $(m^2+9)(m^2+9) = 0$

(Para y_p) (Para y_c)

Para $y_c \rightarrow (m^2+9)=0 \rightarrow m=\pm 3i,\ \alpha=0,\ r=3$

$y_c = C_1\cos(3t)+C_2\sin(3t)$

Para $y_p \rightarrow (m^2+9)=0 \rightarrow m=\pm 3i,\ \alpha=0,\ r=3$

$y_p = At\cos(3t)+Bt\sin(3t)$

$y'_p = A\cos(3t)-3At\sin(3t)+B\sin(3t)+3Bt\cos(3t)$

$y''_p = -3A\sin(3t)-3A\sin(3t)-9At\cos(3t)+3B\cos(3t)+3B\cos(3t)$
$-9Bt\sin(3t) = -6A\sin(3t)-9At\cos(3t)+6B\cos(3t)-9Bt\sin(3t)$

$y'' = -6A\sin(3t)-9At\cos(3t)+6B\cos(3t)-9Bt\sin(3t)$

$9y = 9At\cos(3t)+9Bt\sin(3t)$

$y''+9y = -6A\sin(3t)+6B\cos(3t)$

$-6A\sin(3t)+6B\cos(3t) = 6\sin(3t)$

$-6A=6 \rightarrow A=-1,\quad 6B=0 \rightarrow B=0$

$y_p = -t\cos(3t)+(0)t\sin(3t) \rightarrow y_p = -t\cos(3t)$

$y(t) = C_1\cos(3t)+C_2\sin(3t)-t\cos(3t)$

$y'(t) = -3C_1\sin(3t)+3C_2\cos(3t)-\cos(3t)+3t\sin(3t)$

$1 = C_1\cos(3(0))+C_2\sin(3(0))-(0)\cos(3(0)) \rightarrow \boxed{C_1=1}$

$0 = -3(1)\sin(3(0))+3C_2\cos(3(0))-\cos(3(0))+3(0)\sin(3(0))$

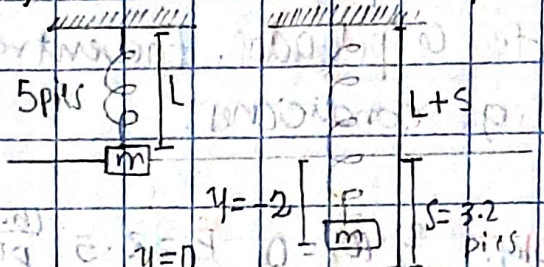
$C_2=\frac{1}{3}$

$y(t) = \cos(3t)+\frac{1}{3}\sin(3t)-t\cos(3t) \rightarrow$ Ec. del movimiento

10) Un objeto que pesa 16 lb se una a un resorte de 5 pies de longitud. En la posición de equilibrio, el resorte mide 8.2 pies. Si el peso se eleva y se suelta del reposo en un punto a 2 pies arriba de la posición de eq., determine los desplazamientos $y(t)$...

$W = 16lb$, $S = 8.2 - 5 = 3.2$ pies

$y(0) = -2$ pies, $y'(0) = 0$



$\beta = 1$, $k = \frac{16lb}{3.2pie} = \frac{5lb}{pies}$

$m = \frac{16lb}{32\frac{pie}{s^2}} = \frac{1}{2}$ slug

$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0 \rightarrow \frac{1}{2}y'' + 1y' + 5y = 0$

$2\left[\frac{1}{2}y'' + y' + 5y\right] = 2[0] \rightarrow y'' + 2y' + 10y = 0$

Ec. auxiliar $\rightarrow m^2 + 2m + 10 = 0 \rightarrow \sqrt{(2)^2 - 4(1)(10)} \rightarrow \sqrt{-36} < 0$

El sistema está subamortiguado (sí oscila).

$m = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm \frac{\sqrt{36}\sqrt{-1}}{2} = -1 \pm 3i$, $\alpha = -1$, $r = 3i$

$m_1 = -1 + 3i$, $m_2 = -1 - 3i$

$y(t) = C_1e^{-t}\cos(3t) + C_2e^{-t}\sin(3t)$

$y'(t) = -C_1e^{-t}\cos(3t) - 3C_1e^{-t}\sin(3t) - C_2e^{-t}\sin(3t) + 3C_2e^{-t}\cos(3t)$

$-2 = C_1e^{-0}\cos(3(0)) + C_2e^{-0}\sin(3(0)) \rightarrow C_1 = -2$

$0 = -(-2)e^{-0}\cos(3(0)) - 3C_1e^{-0}\sin(3(0)) - C_2e^{-0}\sin(3(0)) + 3C_2e^{-0}\cos(3(0))$

$-2 = 3C_2 \rightarrow C_2 = -\frac{2}{3}$

$y(t) = -2e^{-t}\cos(3t) - \frac{2}{3}e^{-t}\operatorname{sen}(3t)$

11) Se encontró experimentalmente que un cuerpo de 4 lb estira un resorte 6 pulgadas. Encuentre la ec. del movi. $y(t)$ considerando las sig. condiciones:

a) $y(0) = 4$ pulg., $y'(0) = 0$, $\beta = 2.5 \frac{\text{lb.s}}{\text{pie}}$.

$w = 4$ lb. $s = 6$ pulgadas. $s = \frac{1}{2}$ pie $\quad k = \frac{F}{s} = \frac{4\text{ lb}}{\frac{1}{2}\text{ pie}} = 8\frac{\text{lb}}{\text{pie}}$, $\quad m = \frac{4\text{ lb}}{32\frac{\text{pie}}{\text{s}^2}} = \frac{1}{8}$ slug

$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0 \longrightarrow \frac{1}{8}y'' + 2.5y' + 8y = 0$

$8\left[\frac{1}{8}y'' + 2.5y' + 8y\right] = 0\,[8] \longrightarrow y'' + 20y' + 64y = 0$

Ec. aux. $\rightarrow m^2 + 20m + 64 = 0$

$m = \frac{-20 \pm \sqrt{(20)^2 - 4(1)(64)}}{2} = -10 \pm \frac{\sqrt{144}}{2} = -10 \pm 6$, $m_1 = -4$, $m_2 = -16$

$y(t) = C_1e^{-16t} + C_2e^{-4t}$, $y'(t) = -16C_1e^{-16t} - 4C_2e^{-4t}$

$\frac{1}{3} = C_1e^{-16(0)} + C_2e^{-4(0)} \rightarrow C_1 = \frac{1}{3} - C_2 \rightarrow C_1 = -\frac{1}{9}$

$0 = -\frac{16}{3} + 16C_2 - 4C_2 \longrightarrow C_2 = +\frac{4}{9}$

$y(t) = \frac{-1}{9}C_1e^{-16t} + \frac{4}{9}C_2e^{-4t}$

c) $y(0)=0$, $y'(0)=\frac{1}{3}\frac{pie}{s}$, $\beta=2\frac{lb.s}{pie}$ Calcular el desplazamiento máx. y el desplazamiento en un largo tiempo.

$y(t)=C_1e^{-\beta t}+C_2te^{-\beta t}$

$y'(t)=-\beta C_1e^{-\beta t}+C_2e^{-\beta t}-\beta C_2te^{-\beta t}$

$0=C_1e^{-\beta(0)}+C_2(0)e^{-\beta(0)} \rightarrow C_1=0$

$\frac{1}{3}=-\beta(0)e^{-\beta(0)}+C_2e^{-\beta(0)}-\beta C_2(t)e^{-\beta(0)} \rightarrow C_2=\frac{1}{3}$

$y(t)=\frac{1}{3}te^{-\beta t}$

Si° $y'(t)=0$ → para el desplazamiento máximo. $\frac{d}{dt}$

$y'(t)=\frac{1}{3}e^{-\beta t}-\frac{\beta}{3}te^{-\beta t} \rightarrow 0=\frac{1}{3}e^{-\beta t}-\frac{\beta}{3}te^{-\beta t}$

$-\frac{1}{3}e^{-\beta t}$

$-\frac{\beta}{3}e^{-\beta t}$ $= t \rightarrow t=\frac{1}{\beta} \rightarrow y\left(\frac{1}{\beta}\right)=\frac{1}{3}\left(\frac{1}{\beta}\right)e^{-\beta\left(\frac{1}{\beta}\right)}=\frac{1}{24}e^{-1}$

$y\left(\frac{1}{\beta}\right)=0.01533$ pie → desplazamiento máximo

Buscando el desplazamiento en un largo tiempo:

$y(\infty)=\frac{1}{3}te^{-\beta t} \rightarrow \frac{1}{3}\lim_{t\to\infty}\frac{t}{e^{-\beta t}} \rightarrow \frac{1}{3}\lim_{t\to\infty}\frac{\infty}{\infty} \rightarrow$ Indefinido

$\frac{\infty}{\infty}$ $\frac{1}{3}\lim_{t\to\infty}\frac{1}{-\beta e^{-\beta t}}=\frac{1}{\infty}=0$

R// El desplazamiento máx. en un largo tiempo es 0.

d) $y(0) = 4$ pulgadas. $y'(0) = 0$, $\beta = 1 \frac{lb \cdot s}{pie}$. Escriba en la forma $y(t) = e^{\alpha t}[C_1 \cos(\omega t) + C_2 \operatorname{sen}(\omega t)]$. Determine los instantes en los que el cuerpo pasa por la posición de equi.

$y(0) = 4 \text{ pug.} \left| \frac{1 \text{ pie}}{12 \text{ pugo.}} \right| = \frac{1}{3}$ pie. $m = \frac{1}{8}$ slug. $K = \frac{8 \text{ lb}}{\text{pie}}$, $\beta = 1$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0 \rightarrow \frac{1}{8}y'' + y' + 8y = 0$$

$$y'' + 8y' + 64y = 0 \rightarrow m^2 + 8m + 64 = 0$$

$$m = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(64)}}{2(1)} = \frac{-8 \pm \sqrt{-192}}{2} = -4 \pm \frac{\sqrt{8^2 \cdot 3}\, i}{2}$$

$$m = -4 \pm \frac{8\sqrt{3}\, i}{2} = -4 \pm 4\sqrt{3}\, i, \quad \alpha = -4, \ \beta = 4\sqrt{3}$$

$$y(t) = C_1 e^{-4t}\cos(4\sqrt{3}\,t) + C_2 e^{-4t}\operatorname{sen}(4\sqrt{3}\,t)$$

$$y'(t) = -4C_1 e^{-4t} - 4\sqrt{3}\,C_1 e^{-4t}\operatorname{sen}(4\sqrt{3}\,t) + 4C_2 e^{-4t}\operatorname{sen}(4\sqrt{3}\,t) + 4\sqrt{3}\,C_2 e^{-4t}\cos(4\sqrt{3}\,t)$$

$$\frac{1}{3} = C_1 \underbrace{e^{-4(0)}}_{\to 1}\cos(4\sqrt{3}(0)) + C_2 \underbrace{e^{-4(0)}}_{\to 1}\operatorname{sen}(4\sqrt{3}(0)) \rightarrow C_1 = \frac{1}{3}$$

$$0 = -4\left(\frac{1}{3}\right)\underbrace{e^{-4(0)}}_{\to 1} - 4\sqrt{3}\left(\frac{1}{3}\right)\underbrace{e^{-4(0)}}_{\to 1}\operatorname{sen}(4\sqrt{3}(0)) - 4C_2\underbrace{e^{-4(0)}}_{\to 1}\operatorname{sen}(4\sqrt{3}(0)) + 4\sqrt{3}\,C_2\underbrace{e^{-4(0)}}_{\to 1}\cos(4\sqrt{3}(0))$$

$$0 = -\frac{4}{3} + 4\sqrt{3}\,C_2 \rightarrow \frac{4}{3} = 4\sqrt{3}\,C_2 \rightarrow C_2 = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$$

$y(t) = \frac{1}{3} e^{-4t} \cos(4\sqrt{3}\,t) + \frac{\sqrt{3}}{9} e^{-4t} \operatorname{sen}(4\sqrt{3}\,t) \rightarrow$ Ec. del mov.

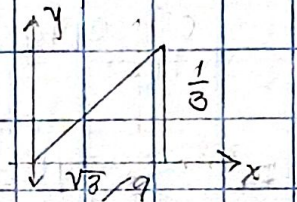
$y(t) = A \operatorname{sen}(4\sqrt{3}\,t + \phi) \rightarrow$ Ec. alternativa

Buscando la amplitud:

$$|A| = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{3}}{9}\right)^2} = \sqrt{\frac{1}{9} + \frac{3}{81}} = \sqrt{\frac{4}{27}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

Para ϕ:

$$\tan\phi = \frac{C_1}{C_2} = \frac{\frac{1}{3}}{\frac{\sqrt{3}}{9}} = \frac{9}{3\sqrt{3}} = \sqrt{3}$$



$\phi = \tan^{-1}(\sqrt{3}) = 1.047$

$\phi = 60° = \frac{\pi}{3}$

$y(t) = \frac{2\sqrt{3}}{9} \operatorname{sen}\left(4\sqrt{3}\,t + \frac{\pi}{3}\right)$; Instantes en los que el cuerpo pasa por la pos. de equ. $y(t) = 0$

$$0 = \frac{2\sqrt{3}}{9} \operatorname{sen}\left(4\sqrt{3}\,t + \frac{\pi}{3}\right) \rightarrow \operatorname{sen}(0) = 0 \rightarrow 0 = 4\sqrt{3}\,t + \frac{\pi}{3}$$

$-\frac{\pi}{3}\left(\frac{1}{4\sqrt{3}}\right) = t \rightarrow$ No puede haber tiempo negativo.

Tomando π

$\pi - \frac{\pi}{3}\left(\frac{1}{4\sqrt{3}}\right) = t \rightarrow t = \frac{\sqrt{3}\,\pi}{12} = 0.3023$ Seg.

Tomando 2π:

$$2\pi - \frac{\pi}{3}\left(\frac{1}{4\sqrt{3}}\right) = t \rightarrow \frac{5\pi}{3}\left(\frac{1}{4\sqrt{3}}\right) = t \rightarrow \left(\frac{5\pi}{12\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = t \rightarrow t = \frac{5\sqrt{3}\,\pi}{36} = 0.756 \text{ Seg}$$

R// Instantes $\rightarrow$ $t_4 = \frac{2\pi - \phi}{\omega} = \frac{2\pi - \frac{\pi}{3}}{4\sqrt{3}} = \frac{\frac{5\pi}{3}}{4\sqrt{3}} = \frac{5\pi}{12\sqrt{3}}$

12) Un peso de 8 lb estira un resorte 2 pies y el sist. se encuentra en un medio que ofrece una resist. numéricamente igual a β veces la veloc. instantánea, donde β es una cte. +. Determinar los valores de β para que el mov. sea:

a) Sobreamortiguado: $\lambda^2 - \omega^2 > 0$

$W = 8\,lb$, $S = 2\,pies$, $K = \frac{W}{S} = \frac{8\,lb}{2\,pies} = 4\,\frac{lb}{pie}$, $m = \frac{8\,lb}{32\,\frac{pie}{s^2}} = \frac{1}{4}\,slug.$

$\frac{1}{4}y'' + \beta y' + 4y = 0 \rightarrow y'' + 4\beta y' + 16y = 0$

$m^2 + 4\beta m + 16 = 0$

$m = \frac{-4\beta \pm \sqrt{(4\beta)^2 - 4(1)(16)}}{2}$ → tomando lo de adentro del radical

$(4\beta)^2 - 4(1)(16) > 0 \rightarrow 16\beta^2 - 64 > 0 \rightarrow \beta^2 > \frac{64}{16}$

$\sqrt{\beta^2} > \sqrt{4} \rightarrow \beta > \pm 2$

Se toma el valor ⊕ ya que β puede tomar valores mayores a 0.

$\therefore \beta > 2 \rightarrow$ R//

b) Críticamente amortiguado. $\lambda^2 - \omega^2 = 0$

$16\beta^2 - 64 = 0 \rightarrow \beta^2 = \frac{64}{16} \rightarrow \sqrt{\beta^2} = \sqrt{4}$

$\beta = \pm 2$, tomando sólo el valor ⊕

$\therefore$ R// $\rightarrow \beta = 2$

c) Subamortiguado: $\lambda^2 - \omega^2 < 0$

16) $\beta^2 - 64 < 0 \rightarrow \beta^2 < 64 \rightarrow \sqrt{\beta^2} = \sqrt{4} \rightarrow \beta < \pm 2$

16

$0 < \beta < 2$

13) Un peso de 4lb se suspende de un resorte cuya cte. $k = 8 \frac{lb}{pie}$...

$w = 4lb$, $k = 8 \frac{lb}{pie}$. $f(t) = 4\cos(8t)$, $\beta = 0$. $y(0) = 0$, $y'(0) = 0$

$s = \frac{F}{k} = \frac{4}{8} = \frac{1}{2}$, $m = \frac{4lb}{32 \frac{pie}{s^2}} = \frac{1}{8}$ slug.

$\frac{1}{8} y'' + 8y = f(t) \rightarrow y'' + 64y = 32\cos(8t)$ $[D^2 + 64]$

$(D^2+64)(D^2+64) = [D^2(32\cos(8t)) + 64(32\cos(8t))]$

$(D^2+64)(D^2+64) = [D(-256\,\mathrm{sen}(8t)) + 2048\cos(8t)]$

$(D^2+64)(D^2+64) = -2048\cos(8t) + 2048\cos(8t)$

$(D^2+64)(D^2+64) = 0$

Ec. auxiliar $\rightarrow$ $(m^2 + 64)(m^2 + 64) = 0$

(m^2+64) para y_p (m^2+64) para y_c

Para $y_c \rightarrow (m^2 + 64) = 0 \rightarrow m^2 = -64 \rightarrow m_1 = 8i$

$y_c = C_1\cos(8t) + C_2\,\mathrm{sen}(8t)$ $m_2 = -8i$

Para $y_p \rightarrow (m^2 + 64) = 0 \rightarrow m_3 = 8i$, $m_3 = -8i$

$y_p = At\cos(8t) + Bt\,\mathrm{sen}(8t)$

$y'_p = A\cos(8t) - 8At\,\mathrm{sen}(8t) + B\,\mathrm{sen}(8t) + 8Bt\cos(8t)$

$y''_p = -8A\,\mathrm{sen}(8t) - 8A\,\mathrm{sen}(8t) - 64At\cos(8t) + 8B\cos(8t)$

$+ 8B\cos(8t) - 64Bt\,\mathrm{sen}(8t)$

Arimetrix

$y'' = -16A\sin(8t) - 64At\cos(8t) + 16B\cos(8t) - 64Bt\sin(8t)$

$64y = 64At\cos(8t) + 64Bt\sin(8t)$

$y'' + 64y = -16A\sin(8t) + 16B\cos(8t)$

$-16A\sin(8t) + 16B\cos(8t) = 32\cos(8t)$

$-16A = 0 \rightarrow A = 0$; $16B = 4 \rightarrow B = \frac{1}{4}$

$y_p = \frac{1}{4}t\sin(8t)$; $y(t) = C_1\cos(8t) + C_2\sin(8t) + \frac{1}{4}t\sin(8t)$

$y'(t) = -8C_1\sin(8t) + 8C_2\cos(8t) + \frac{1}{4}\sin(8t) + \frac{1}{4}t\cos(8t)$

$0 = C_1\cos(8(0)) + C_2\sin(8(0)) + \frac{1}{4}\sin(8(0))(0)$, $C_1 = 0$

$0 = -8(0)\sin(8(0)) + 8C_2\cos(8(0)) + \frac{1}{4}\sin(8(0)) + \frac{1}{4}(0)\cos(8(0))$

$C_2 = 0$

$\therefore y(t) = \frac{1}{4}t\sin(8t)$